Exact solution of a reaction-diffusion model with particle-number conservation

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(Received 24 June 2004; published 22 November 2004)

We analytically investigate a one-dimensional branching-coalescing model with reflecting boundaries in a canonical ensemble where the total number of particles on the chain is conserved. Exact analytical calculations show that the model has two different phases which are separated by a second-order phase transition. The thermodynamic behavior of the canonical partition function of the model has been calculated exactly in each phase. Density profiles of particles have also been obtained explicitly. It is shown that the exponential part of the density profiles decays on three different length scales which depend on the total density of particles.

DOI: 10.1103/PhysRevE.70.056121 PACS number(s): 05.70.Ln, 05.20.−y, 02.50.Ey, 05.70.Fh

I. INTRODUCTION

Recently much attention has been paid to the study of shocks in one-dimensional reaction-diffusion models [1–4]. The shocks are defined as discontinuities in the space dependence of density of particles in the system and behave as collective excitations in the system. They can be characterized by their position which performs a random walk. The best known example in which shock can appear is the asymmetric simple exclusion process (ASEP) with open boundaries [5]. The mathematical relevance of the ASEP is that it is a discrete version of the Burgers equation in an appropriate scaling limit. The ASEP contains one class of particles (first-class particles) which can be injected and extracted from the boundaries of a one-dimensional chain while hopping in the bulk with asymmetric rates. The ASEP has several applications to realistic systems. For instance, it can be considered as a simple model for traffic flow [6].

There are different ways to provoke a shock in one-dimensional reaction-diffusion models. One can consider the ASEP on a closed chain in the presence of a second-class particle. Compared to first-class particles, second-class particles move very slowly. In [7–12] the shape of the shock is calculated as seen from a second-class particle. Another method is to introduce a slow link in the system [13]. The first-class particles cross this link with a smaller crossing rate than that of the other links in the system. In this case the width of the shock as a function of the length of the system L scales as $L^{1/3}$ or $L^{1/2}$ depending on whether the density of particles is equal to $\frac{1}{2}$ or not [14]. Shocks have also been observed in the ASEP with creation and annihilation of particles in the bulk of the system [15,16].

In a recent paper we have numerically studied shocks in a spatially asymmetric one-dimensional branching-coalescing model with reflecting boundaries in a canonical ensemble [17]. In this model the particles diffuse, coagulate, and decoagulate on a lattice of length L; however, the total number of the particles is kept fixed. It is predicted that the model has two different phases and in one phase the density profile of the particles has a shock structure. We have confirmed our numerical results by using the Yang-Lee theory of phase transitions [18] which has recently been shown to be applicable to the study of the critical behaviors of out-of-equilibrium systems [19,20]. In the present work we will show that by working in the canonical ensemble, the model is exactly solvable in the sense that the thermodynamic limit of physical quantities can be calculated exactly. The canonical partition function of the model defined as the sum over stationary-state weights can also be calculated exactly. By applying the Yang-Lee theory we can calculate the line of the partition function zeros and, therefore, spot the transition point. The order of the transition can also be identified by investigating the density of these zeros near the critical point. We will also obtain the exact expressions for the density profile of the particles on the chain in the thermodynamic limit. This paper is organized as follows: In Sec. II we will define the model and introduce the mathematical preliminaries. In Sec. III we will calculate the canonical partition function of our model using a matrix product formalism and find its behavior in the thermodynamic limit. We will also find the line of canonical partition function zeros to confirm our numerical results in [17]. In Sec. IV we will calculate the density profile of the particles on the chain in each phase. In the last section we will discuss the results and compare them with the case where the total number of particles is not conserved.

II. MODEL: MATHEMATICAL PRELIMINARIES

In this section we will briefly review the definition of the model and also define its grand canonical partition function. We will then calculate the canonical partition function of the model explicitly. The model consists of one class of particles which diffuse on a one-dimensional chain of length L. Whenever two of these particles meet, they can coagulate to a single particle. In the same way, a single particle can decoagulate into two particles. There is no particle input or output at the boundaries. The reaction rules between two consecutive sites i and $i+1$ on the chain are explicitly as follows:

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$$\emptyset + A \to A + \emptyset \text{ with rate } q,$$

$$A + \emptyset \to \emptyset + A \text{ with rate } q^{-1},$$

$$A + A \to A + \emptyset \text{ with rate } q,$$

$$A + A \to \emptyset + A \text{ with rate } q^{-1},$$

$$\emptyset + A \to A + A \text{ with rate } \Delta q,$$

$$A + \emptyset \to A + A \text{ with rate } \Delta q^{-1}, \quad (1)$$

in which A and $\emptyset$ stand for the presence of a particle and a hole, respectively. As can be seen, the parameter q determines the asymmetry of the model. For $q>1$ ($q<1$) the particles have a tendency to move in the leftward (rightward) direction. For any q the model is also invariant under the following transformations:

$$q \to q^{-1}, \quad i \to L - i + 1, \quad (2)$$

in which i is a given site on the chain. One should also note that the rules (1) do not conserve the number of particles and therefore the model should be studied in a grand canonical ensemble. The model without particle number conservation has already been studied both using empty interval method (EIM) and matrix product formalism (MPF) [22,23]. It turns out that the model has two different phases in this case: two exponential phases which are called the height-density and the low-density phases. On the coexistence line of these phases the density of the particles on the chain has a linear profile. It is known that the phase diagram of the ASEP contains a first-order phase transition line where the injection and extraction rates are equal and less than $\frac{1}{2}$. Along this line the density profile of particles is linear which is a consequence of superposition of states where a shock between a low-density region and a height-density region is present at an arbitrary position [5,21]. As we will see the linear profile in the present model can also be interpreted as a sign for the existence of shocks.

In order to study the shocks we restrict the total number of particles on the chain to be M so that their total density is always equal to $\rho = M/L$. This means that we are working in a canonical ensemble. The stationary-state probability distribution function can be calculated using the MPF as follows: We assign two noncommuting operators D and E to a particle and a hole, respectively. Now the probability for occurring any configuration $\{\tau\} = \{\tau_1, \ldots, \tau_L\}$ in the steady state with exactly M particles can be obtained from

$$P(\{\tau\}) = \frac{\delta(M - \sum_{i=1}^{L} \tau_i)}{Z_{L,M}} \langle W | \prod_{i=1}^{L} [\tau_i D + (1-\tau_i) E] | V \rangle, \quad (3)$$

in which $\tau_i = 1$ if the site i is occupied by particles and $\tau_i = 0$ if it is empty. The normalization factor $Z_{L,M}$ in Eq. (3), which will be called the canonical partition function of the model hereafter, should be obtained from the fact that $\sum_{\{\tau\}} P(\{\tau\}) = 1$. It can be written as

$$Z_{L,M} = \sum_{\{\tau\}} \delta\left(M - \sum_{i=1}^{L} \tau_i\right) \langle W | \prod_{i=1}^{L} [\tau_i D + (1-\tau_i) E] | V \rangle. \quad (4)$$

The Dirac δ function in Eqs. (3) and (4) guarantees the total number of particles to be M in the steady state. In order to have a stationary probability distribution, the operators D and E besides the vectors $|V\rangle$ and $\langle W|$ should satisfy the quadratic algebra [23]

$$[E, \overline{E}] = 0,$$

$$\overline{E}D - E\overline{D} = q(1+\Delta)ED - \frac{1}{q}DE - \frac{1}{q}D^2,$$

$$\overline{D}E - D\overline{E} = -qED + \frac{1+\Delta}{q}DE - qD^2,$$

$$\overline{D}D - D\overline{D} = -q\Delta ED - \frac{\Delta}{q}DE + \left(q + \frac{1}{q}\right)D^2,$$

$$\langle W|\overline{E} = \langle W|\overline{D} = 0, \ \overline{E}|V\rangle = \overline{D}|V\rangle = 0. \quad (5)$$

The operators $\overline{D}$ and $\overline{E}$ are auxiliary operators and do not enter into calculating Eqs. (3) and (4). The following four-dimensional representation has been found for the algebra (5) [23]:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Delta}{1+\Delta} & \frac{\Delta}{1+\Delta} & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} a \\ 0 \\ q^2 \\ q^2 - 1 \end{pmatrix},$$

$$E = \begin{pmatrix} q^{-2} & q^{-2} & 0 & 0 \\ 0 & \frac{1}{1+\Delta} & \frac{1}{1+\Delta} & 0 \\ 0 & 0 & 1 & q^2 \\ 0 & 0 & 0 & q^2 \end{pmatrix}, \quad |W\rangle = \begin{pmatrix} 1 - q^2 \\ 1 \\ 0 \\ b \end{pmatrix}, \quad (6)$$

in which a and b are arbitrary constants and $|W\rangle$ is simply a transpose of $\langle W|$. The matrix representations for $\overline{D}$ and $\overline{E}$ are also given in [23]. Using Eqs. (6) one can calculate the steady-state weight of any given configuration.

It turns out that the direct calculation of Eq. (4) is not always an easy task; therefore, we define the grand canonical partition function which can easily be calculated:

$$Z_L(\xi) = \sum_{\{\tau\}} \langle W | \prod_{i=1}^{L} [\tau_i \xi D + (1-\tau_i) E] | V \rangle = \sum_{M=0}^{L} \xi^M Z_{L,M}, \quad (7)$$

in which ξ is the fugacity associated with the particles. The total density of particles ρ should then be fixed by the fugacity of them through the equation

$$\rho = \lim_{L\to\infty} \frac{\xi}{L}\frac{\partial}{\partial \xi} \ln Z_L(\xi). \tag{8}$$

One can expect that each value of the fugacity ξ corresponds to each value of the total density. In this case, the density-fugacity relation (8) is invertible and the equivalence of the canonical and grand canonical ensemble holds. After calculating the grand canonical partition function (7), one can invert the series to calculate the canonical partition function using

$$Z_{L,M} = \frac{1}{2\pi i}\int_C d\xi \frac{Z_L(\xi)}{\xi^{M+1}}, \tag{9}$$

where C is a contour which encircles the origin anticlockwise. For our model, however, there appears a situation where the equivalence of ensembles fails in a special region in the parameters space. There is the place where the shocks appear in the system.

As an important physical quantity one can study the density profile of particles on the chain in the canonical ensemble; nevertheless, the calculation of the density profile of the particles is much more easily done in the grand canonical ensemble. Let us define the unnormalized average particle number at site i in the grand canonical ensemble as

$$\langle \rho_i \rangle_L^{(u)}(\xi) = \sum_{\{\tau\}} \langle W| \prod_{j=1}^{i-1} [\tau_j \xi D + (1-\tau_j)E]\xi D \times \prod_{j=i+1}^{L} [\tau_j \xi D + (1-\tau_j)E]|V\rangle. \tag{10}$$

We will then translate the results in the grand canonical ensemble into those in the canonical ensemble using the formula

$$\langle \rho_i \rangle_{L,M}^{(u)} = \frac{1}{2\pi i}\int_C d\xi \frac{\langle \rho_i \rangle_L^{(u)}(\xi)}{\xi^{M+1}}. \tag{11}$$

As in Eq. (9) the contour C in Eq. (11) encircles the origin anticlockwise. In Eqs. (10) and (11) the superscript (u) means that it is an unnormalized quantity. The normalized average particle number at site i should be obtained from $\langle \rho_i \rangle = \langle \rho_i \rangle_{L,M}^{(u)} / Z_{L,M}$.

III. CANONICAL AND GRAND CANONICAL PARTITION FUNCTIONS

In this section we will calculate the grand canonical partition function of the model explicitly and then using Eq. (9) one can obtain the canonical partition function by applying the steepest decent method. The grand canonical partition function of this model can easily be calculated using Eq. (7) and is simply given by

$$Z_L(\xi) = \langle W|(\xi D + E)^L|V\rangle = \langle W|C^L|V\rangle, \tag{12}$$

in which we have defined $C \coloneqq \xi D + E$. The matrix representations for the operators D and E are given by Eqs. (6). After some algebra we find

$$Z_L(\xi) = Z_L^{(1)}(\xi) + Z_L^{(2)}(\xi) + Z_L^{(3)}(\xi) + Z_L^{(4)}(\xi), \tag{13}$$

in which

$$Z_L^{(1)}(\xi) = \left[\frac{-q^4\Delta\xi^2}{[q^2-(1+\xi\Delta)][q^2(1+\xi\Delta)-1]}\right](1+\xi\Delta)^L, \tag{14}$$

$$Z_L^{(2)}(\xi) = \left[\frac{q^4(q^2-1)(1+\xi\Delta)}{(q^2+1)[q^2-(1+\xi\Delta)][q^2(1+\Delta)-(1+\xi\Delta)]}\right]q^{2L}, \tag{15}$$

$$Z_L^{(3)}(\xi) = \left[\frac{-q^4(q^2-1)(1+\xi\Delta)}{(q^2+1)[(1+\Delta)-q^2(1+\xi\Delta)][1-q^2(1+\xi\Delta)]}\right]q^{-2L},0 \tag{16}$$

$$Z_L^{(4)}(\xi) = \left[\frac{q^4\Delta(\xi-1)^2}{[q^2(1+\Delta)-(1+\xi\Delta)][q^2(1+\xi\Delta)-(1+\Delta)]}\right]\left(\frac{1+\xi\Delta}{1+\Delta}\right)^L. \tag{17}$$

Because of the symmetry of the model (2), one will only need to study either the case $q>1$ or $q<1$. We will consider the case $q>1$ hereafter, and the results for the case $q<1$ can easily be obtained by applying the transformations (2). Obviously for $q>1$ we have $q^2>q^{-2}$. On the other hand, since $\Delta,\ \xi>0$, we always have $(1+\xi\Delta)>[(1+\xi\Delta)/(1+\Delta)]$. Now two different cases can be distinguished: We will either have $1<q<\sqrt{1+\xi\Delta}$ or $1<\sqrt{1+\xi\Delta}<q$. For these two cases the

asymptotic behaviors of the grand canonical partition function (13) can be obtained in the large-system-size limit $L \to \infty$:

$$Z_L(\xi) \simeq \begin{cases} Z_L^{(1)}(\xi), & 1 < q < \sqrt{1+\xi\Delta}, \\ Z_L^{(2)}(\xi), & 1 < \sqrt{1+\xi\Delta} < q. \end{cases} \tag{18}$$

For a fixed total density of particles ρ (which means fixed ξ) and Δ, the phase transition occurs at $q_c = \sqrt{1+\xi\Delta}$. Now one can easily calculate the canonical partition function of the system in these phases using Eq. (9). By using Eq. (8) for the first phase the condition $1 < q < \sqrt{1+\xi\Delta}$ translates to $1 < q < 1/(\sqrt{1-\rho})$ and the canonical partition function, which is given by

$$Z_{L,M}^{(I)} \simeq \frac{1}{2\pi i}\int_C d\xi \frac{Z_L^{(1)}(\xi)}{\xi^{M+1}}, \tag{19}$$

can readily be calculated by applying the steepest decent method. We find

$$Z_{L,M}^{(I)} \simeq \frac{q^4 \Delta^{M-1} \rho^{3/2-M}(1-\rho)^{M-L-1/2}}{[1-(1-\rho)q^2][q^2-(1-\rho)]}, \quad 1 < q < \frac{1}{\sqrt{1-\rho}}. \tag{20}$$

For the second phase the condition $1 < \sqrt{1+\xi\Delta} < q$ translates to $1 < 1/(\sqrt{1-\rho}) < q$. We have also

$$Z_{L,M}^{(II)} \simeq \frac{1}{2\pi i}\int_C d\xi \frac{Z_L^{(2)}(\xi)}{\xi^{M+1}}. \tag{21}$$

Keeping in mind that the contour of the integral above is a unit circle and that its integrand has two poles, one of which, $\xi_1 = (q^2-1)/\Delta$, is smaller than unity and the other $\xi_2 = \xi_1 + q^2$ is larger than unity, one can easily calculate Eq. (21) using the steepest decent method. We find

$$Z_{L,M}^{(II)} \simeq \frac{q^{4+2L}\Delta^{M-1}}{(q^2+1)(q^2-1)^M}, \quad 1 < \frac{1}{\sqrt{1-\rho}} < q. \tag{22}$$

It can be seen that for a fixed density ρ the transition point $q_c = 1/(\sqrt{1-\rho})$ does not depend on Δ. This has already been predicted in [17]. For the case $q < 1$ the transition point is found to be $q'_c = \sqrt{1-\rho}$ which agrees again with our predictions in [17].

In [17] we have estimated the roots of the canonical partition function $Z_{L,M}$ as a function q both for $q > 1$ and $q < 1$. From there we were able to find the transition points. Let us now calculate the line of the canonical partition function zeros of the model in the complex q plane for $q > 1$. Defining the extensive part of the free energy as

$$g = \lim_{L,M \to \infty} \frac{1}{L} \ln Z_{L,M}, \tag{23}$$

one can calculate the line of canonical partition function zeros from

$$\text{Re}\, g^{(I)} = \text{Re}\, g^{(II)}, \tag{24}$$

in which $g^{(I)}$ and $g^{(II)}$ are the free energy functions in the first and the second phases, respectively. Using Eqs. (20) and (22)–(24), we find, in the thermodynamic limit ($L, M \to \infty, \rho = M/L$),

$$\frac{u^2+v^2}{[(u^2-v^2-1)^2+(2uv)^2]^{\rho/2}} = \frac{(1-\rho)^{\rho-1}}{\rho^\rho}, \tag{25}$$

in which we have defined $u := \text{Re}(q)$ and $v := \text{Im}(q)$. It can easily be verified that Eq. (25) intersects the positive real q axis at $u_c = 1/(\sqrt{1-\rho})$. As can be seen, Eq. (25) is exactly the one that we had obtained in [24] for the same model with the left boundary open and conservation of total number of particles. In [24] we had also found that the density of canonical partition function zeros as a function of q drops to zero near the critical point. This indicates that a second-order phase transition takes place at the critical point. We have checked that the density of canonical partition function zeros in the present model also approaches to zero near the critical points q_c and q'_c.

For $q < 1$ we should only change $q \to q^{-1}$ which means $u \to u/(u^2+v^2)$ and $v \to -v/(u^2+v^2)$ in Eq. (25). In this case the line of canonical partition function zeros intersects the positive real q axis at $u'_c = \sqrt{1-\rho}$. It is not difficult to check that in the thermodynamic limit the numerical estimates for the canonical partition function zeros obtained in [17] lie exactly on Eq. (25) and its counterpart for $q < 1$.

IV. DENSITY PROFILE OF PARTICLES

Now we consider the average particle number at each site. As for the partition functions, it turns out that the calculation of density profile of the particles in the grand canonical ensemble is much easier than that in the canonical ensemble; therefore, we will first calculate (10) and then translate out results into the canonical ensemble using Eq. (11). The unnormalized average particle number at site i in the grand canonical ensemble (10) can also be written as

$$\langle \rho_i \rangle_L^{(u)}(\xi) = \langle W | C^{i-1} \xi D C^{L-i} | V \rangle, \tag{26}$$

in which $C := \xi D + E$. Now one can use the matrix representation (6) to calculate Eq. (26). After some algebra we find

$$\begin{aligned} \langle \rho_i \rangle_L^{(u)}(\xi) = {} & u_1(\xi) q^{2L-4i+2} \\ & + u_2(\xi) q^{2-2i}(1+\xi\Delta)^{L-i} + u_3(\xi) q^{2-2i}\left(\frac{1+\xi\Delta}{1+\Delta}\right)^{L-i} \\ & + u_4(\xi) q^{2L-2i}(1+\xi\Delta)^{i-1} + u_5(\xi) q^{2L-2i}\left(\frac{1+\xi\Delta}{1+\Delta}\right)^{i-1} \\ & + u_6(\xi)(1+\xi\Delta)^{L-1} + u_7(\xi)\left(\frac{1+\xi\Delta}{1+\Delta}\right)^{L-1}, \end{aligned} \tag{27}$$

in which we have defined

$$u_1(\xi) = \frac{q^4(q^2-1)^2\xi\Delta^2[\xi(2+\xi\Delta)-1](q^2-\xi\Delta-1)^{-1}}{[q^2(1+\Delta)-\xi\Delta-1][q^2(1+\xi\Delta)-1][q^2(1+\xi\Delta)-\Delta-1]}, \tag{28}$$

$$u_2(\xi) = \frac{-q^2(q^2-1)\xi^2\Delta}{(q^2-\xi\Delta-1)[q^2(1+\xi\Delta)-1]}, \tag{29}$$

$$u_3(\xi) = \frac{q^2(q^2-1)(\xi-1)\xi\Delta}{[q^2(1+\Delta)-\xi\Delta-1][q^2(1+\xi\Delta)-\Delta-1]}, \tag{30}$$

$$u_4(\xi) = \frac{q^4(q^2-1)\xi^2\Delta}{(q^2-\xi\Delta-1)[q^2(1+\xi\Delta)-1]}, \tag{31}$$

$$u_5(\xi) = \frac{-q^4(q^2-1)(\xi-1)\xi\Delta}{[q^2(1+\Delta)-\xi\Delta-1][q^2(1+\xi\Delta)-\Delta-1]}, \tag{32}$$

$$u_6(\xi) = \frac{-q^4\xi^3\Delta^2}{(q^2-\xi\Delta-1)[q^2(1+\xi\Delta)-1]}, \tag{33}$$

$$u_7(\xi) = \frac{q^4(\xi-1)^2\xi\Delta^2}{(1+\Delta)[q^2(1+\Delta)-\xi\Delta-1][q^2(1+\xi\Delta)-\Delta-1]}. \tag{34}$$

The asymptotic behaviors of Eq. (27) can now be distinguished for the two mentioned cases. For the first case where $1<q<\sqrt{1+\xi\Delta}$ the leading terms are the second, the fourth, and the sixth terms in Eq. (27). Now using Eqs. (11) and (20) one can calculate the average particle number of the particles at site i in the canonical ensemble by applying the steepest decent method. In the thermodynamic limit the result is

$$\langle\rho_i\rangle = \rho + (q^2-1)[e^{-i/\xi_1} - (1-\rho)e^{-(L-i)/\xi_2}],\quad 1<q<\frac{1}{\sqrt{1-\rho}}, \tag{35}$$

in which the correlation lengths are $\xi_1=|\ln[(1-\rho)/q^2]|^{-1}$ and $\xi_2=|\ln[q^2(1-\rho)]|^{-1}$. For a plot of this profile see Fig. 2 in [17]. In the second case where $1<\sqrt{1+\xi\Delta}<q$ the leading terms are the first and fourth terms in Eq. (27). Using numerical calculations we had predicted in [17] that the density profile of the particles in this phase is a shock in the bulk of the chain while it increases exponentially near the left boundary for $q>1$. The density of the particles in the high-density region of the shock is equal to $\rho_{high}=1-q^{-2}$ while in the low-density region, it is zero $\rho_{low}=0$. One can easily calculate the share of the first term in to the density profile of the particles in the canonical ensemble using Eq. (11). By applying the steepest decent method one finds $(1-q^{-2})q^{2-4i}$. In order to calculate the share of the fourth term in Eq. (27) in the grand canonical ensemble we use the following procedure: When L is large, the average density profile can be described by a continuous function $\rho(x)$ in terms of the rescaled variable $x=i/L$ where $0\leqslant x\leqslant 1$. By using Eq. (11) for the fourth term in Eq. (27) we find that the derivative of $\rho(x)$ has the following form:

$$\frac{d}{dx}\rho(x) \simeq \rho_0 \exp[LF(x)], \tag{36}$$

where

$$F(x) = -x\ln q^2 + x\ln\frac{x}{x-\rho} - \rho\ln\frac{\rho}{\Delta(x-\rho)}. \tag{37}$$

The constant ρ_0 in Eq. (36) is determined by the fact that

$$\int_0^1 \frac{d}{dx}\rho(x)dx = \rho_{\text{low}} - \rho_{\text{high}} = q^{-2} - 1. \tag{38}$$

It turns out that the function $F(x)$ has a maximum value at $x_0=\rho/(1-q^{-2})$. One can expand $F(x)$ around x_0 up to the second order and approximate Eq. (36) with a Gaussian and find

$$\frac{d}{dx}\rho(x) \simeq -\sqrt{\frac{L}{2\pi\rho q^{-2}}}(1-q^{-2})^2 \times\exp\left(-L\frac{(1-q^{-2})^2(x-x_0)^2}{2\rho q^{-2}}\right). \tag{39}$$

By integrating Eq. (39) the average particle number at site i in the canonical ensemble for $1<1/(\sqrt{1-\rho})<q$ is found to be

$$\langle\rho_i\rangle = (1-q^{-2})q^2 e^{-i/\xi_3} + \frac{1-q^{-2}}{2} \times \mathrm{erfc}\left[\sqrt{\frac{L}{2\rho q^{-2}}}(1-q^{-2})\left(\frac{i}{L} - \frac{\rho}{1-q^{-2}}\right)\right], \quad (40)$$

$$1 < \frac{1}{\sqrt{1-\rho}} < q,$$

in which the exponential part drops with the length scale $\xi_3 = |\ln q^4|^{-1}$ and erfc$(\cdots)$ is the complementary error function. As can be seen from Eq. (40) the average particle number at site i far from the left boundary is an error function interpolating between the low-density and high-density regions with width scaling as $\sqrt{L}$. For a plot of this profile see Fig. 2 in [17].

V. CONCLUDING REMARKS

In this paper we studied a one-dimensional asymmetric branching-coalescing model with reflecting boundaries in a canonical ensemble where the total number of particles is a constant. This model has already been studied in the literature in a grand canonical ensemble where the total number of particles on the chain is not fixed and can vary between 0 and 1 (see [22,23] and references therein).

Without particle number conservation the parameter Δ, which is the ratio of branching to coalescing rates, governs the average density of particles on the chain. In this case the phase diagram of the model consists of two phases: a high-density phase and a low-density phase. In the height-density phase the density profile of the particles has an exponential behavior with two different correlation lengths $|\ln[q^2/(1+\Delta)]|^{-1}$ and $|\ln[q^2(1+\Delta)]|^{-1}$. In the low-density phase the density profile of the particles has also an exponential behavior; however, with the length scales $|\ln(q^4)|^{-1}$ and $|\ln[q^2/(1+\Delta)]|^{-1}$. On the coexistence line between these two phases the density profile of the particles has a linear decay in one end of the chain while it has an exponential decay in the other end of the chain with the length scale $|\ln(q^4)|^{-1}$.

In the canonical ensemble the total density of particles on the chain is controlled by the parameter ρ instead of Δ. With particle number conservation it turns out that for $q > 1$ the model has two different phases: an exponential phase and a shock phase. In the exponential phase the density profile of the particles has an exponential behavior with two length scales $|\ln[q^2/(1-\rho)]|^{-1}$ and $|\ln[q^2(1-\rho)]|^{-1}$. In the shock phase the density profile of the particles drops exponentially near the left boundary with the length scale $|\ln(q^4)|^{-1}$. In the bulk of the chain the density profile of the particles is an error function with an interface which extends over a region of width $\sqrt{L}$.

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